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RF-34137

M. Sc. - I Examination

April / May – 2010

Mathematics : Paper - 401

(Real Analysis)

Time : **3** Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :

☛ **M. Sc. - 1**

Name of the Subject :

☛ **MATHEMATICS - 401**

☛ Subject Code No. : **3 4 3 7** ☛ Section No. (1, 2,.....) : **NIL**

Seat No. :

Student's Signature

(2) Attempt all questions.

(3) Figures on the right indicate full marks of the question.

(4) Follow usual notations and conventions.

Q.1 (a) Define a measurable set and prove that the collection M of measurable sets is a σ-algebra of sets. 5

(b) Let A be any set and let $E_1, E_2, E_3, E_4, \ldots\ldots, E_n$ be a finite sequence of measurable sets then prove that 5

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i).$$

(c) If E_1 and E_2 are measurable sets then prove that 4

$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

OR

Q.1 (a) Let E be a given set. Then prove that the following statements are equivalent. 5

(i) E is measurable.

(ii) Given $\in > 0$, there is an open set $O \supset E$ with $m^*(O \sim E) < \in$.

(iii) There is a G in G_δ with $E \subset G, m^*(G \sim E) = 0$.

(b) Let $\langle E_i \rangle$ be a sequence of measurable sets. Then prove that $m(\cup E_i) \leq \sum mE_i$. If the sets E_n are pairwise disjoint, then $m(\cup E_i) = \sum mE_i$. 5

(c) If f is a continuous function defined on a measurable domain then prove that f is a measurable function. 4

Q.2 (a) If f is a measurable function then prove that $\inf_{f\le\psi}\int\psi(x)dx=\sup_{f\ge\phi}\int\phi(x)dx$ for all simple functions ϕ and ψ. 5

(b) Let $\langle F_n\rangle$ be a sequence of measurable functions with the same domain of definition then prove that the functions $\sup\{f_1,f_2,....,f_n\}, \inf\{f_1,f_2,....,f_n\}, \sup_n f_n, \inf_n f_n, \overline{\lim} f_n,$ and $\underline{\lim} f_n$ are all measurable. 5

(c) Let u_n be a sequence of nonnegative measurable functions and let $f=\sum_{n=1}^{\infty}u_n$ then prove that $\int f=\sum_{n=1}^{\infty}\int u_n$. 4

OR

Q.2 (a) If f and g are bounded measurable functions defined on a set E of finite measure, then prove that 5

(i) $\int_E(af+bg)=a\int_E f+b\int_E g$

(ii) If $A\le f(x)\le B$, then $AmE\le\int_E f\le BmE$

(iii) If A and B are disjoint measurable sets of finite measure, then $\int_{A\cup B}f=\int_A f+\int_B f$.

(b) Let $\langle f_n\rangle$ be an increasing sequence of nonnegative measurable functions, and if $f=\lim f_n$ a.e. Then prove that $\int f=\lim\int f_n$. 5

(c) Show that if $f(x)=\begin{cases}3\,; x \text{ is irrational}\\ 4\,; x \text{ is rational}\end{cases}$ for $x\in[a,b]$ then $f\notin R[a,b]$. 4

Q.3 (a) Prove that a function f is of bounded variation on $[a,b]$ if and only if f is the difference of two monotone real valued function on $[a,b]$. 5

(b) If f is integrable on $[a,b]$ and $F(x)=F(a)+\int_a^x f(t)dt$ then prove that $F'(x)=f(x)$, for almost all x in $[a,b]$. 5

(c) Let f be a function defined by $f(x)=|x|$ then find D^+, D_+, D^- and D_- for $x=0$. 4

OR

Q.3 (a) If f is absolutely continuous on $[a,b]$ then prove that it is of bounded variation on $[a,b]$. 5

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continuous. 5

(c) Show that if $a\le c\le b$ then $T_a^b=T_a^c+T_c^b$ and hence $T_a^b\ge T_a^c$. 4

Q.4 (a) If f and g are in L^p with $1 \le p \le \infty$ then prove that so is $f+g$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$. 5

(b) Let ϕ be a convex function on $(-\infty, \infty)$ and f and integrable function on $[0,1]$. Then prove that $\int \phi(f(t))\, dt \ge \phi\left[\int f(t)\, dt\right]$. 5

(c) Let $1 \le p < \infty$ and a, b, t are nonnegative then prove that $(a+tb)^p \ge a^p + ptba^{p-1}$. 4

OR

Q.4 (a) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable. 5

(b) If μ is a complete measure and f is a measurable function then prove that $f = g$ a.e. implies g measurable. 5

(c) Define a measurable function and prove that a constant function is measurable. 4

Q.5 (a) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then prove that there is a unique measurable extended real valued function f on X such that $f \le \alpha$ on B_α and $f \ge \alpha$ on $X \sim B_\alpha$. 5

(b) Let $((A_i \times B_i))$ be a countable disjoint collection of Measurable rectangles whose union is measurable rectangle $A \times B$. Then prove that $\lambda(A \times B) = \sum \lambda(A_i \times B_i)$ 5

(c) Prove that the Lebesgue measure is invariant under translation modulo 1. 4

OR

Q.5 (a) Prove that the set function μ^* is an outer measure. 5

(b) Let (X , B) be a measurable space and (μ_n) be a sequence of measures on B that converges set wise to a measure μ. Let $\langle f_n \rangle$ and $\langle g_n \rangle$ be two sequences of measurable functions that converge point wise to f and g. Suppose that $|f_n| \le g_n$ and that $\lim \int g_n d\mu_n = \int g d\mu < \infty$ then prove that $\lim \int f_n \, d\mu_n = \int f \, d\mu$. 5

(c) Define Jorden Decomposition and prove its existence. 4
